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Recursive Formulas on Free Distributive Lattices

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ABSTRACT

The paper is concerned with the problem of computing the number of elements of a free distributive lattice, and with related problems. A recursive formula, using the order of a free distributive lattice with one less generators, is obtained to compute the number of elements of any principal filter of the lattice, in particular the lattice itself. Recursive formulas for the fixed points of the antiorder isomorphism lead easily to the fact that when the number of generators is even the number of elements of the lattice is even. For basic properties of free distributive lattices we refer the reader to [1].

1. DEFINITIONS

We shall call distributive lattice with 0 and 1, a set R with two operations, greatest lower bound ($\wedge$) and lowest upper bound ($\vee$), such that:

(1) $a \wedge b = b \wedge a, \qquad a \wedge (b \wedge c) = (a \wedge b) \wedge c,$

(2) $a \vee b = b \vee a, \qquad a \vee (b \vee c) = (a \vee b) \vee c,$

(3) $a \wedge (b \wedge a) = a,$

(4) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$

(5) $1 \wedge a = 0 \vee a = a.$

The lattice is free with a set $G \subset R$ of free generators if and only if for any distributive lattice K and any mapping $h : G \to K$ there exists a homomorphism $h' : R \to K$ such that for $g \in G \quad h(g) = h'(g)$.

We shall denote by $\mathrm{FD}(n)$ the free distributive lattice with n free generators.

2. TWO CANONICAL PROJECTIONS FROM $\mathrm{FD}(n)$ INTO $\mathrm{FD}(n-1)$

Let $\{g_i\} = G$, the set of free generators of $\mathrm{FD}(n)$.
Define

$$k_j : G \to \mathrm{FD}(n-1)$$

by $k_j(g_j) = 0$, $k_j(g_k) = g_k$, for $k \neq j$, and extend it to homomorphism from FD(n) into FD($n - 1$). Similarly, define

$$l_j : G \to \mathrm{FD}(n-1)$$

by $l_j(g_j) = 1$, $l_j(g_k) = g_k$, $k \neq j$, and extend it to a homomorphism.

It is clear from the definition that, actually, k_j and l_j are homomorphisms from FD(n) onto FD($n - 1$), and also $l_j(x) \geqslant x \geqslant k_j(x)$ for every $x \in \mathrm{FD}(n)$. For $x \in \mathrm{FD}(n-1)$ (g_j excluded) $l_j(x) = k_j(x) = x$. We shall call these two homomorphisms the j-th upper (l_j) and lower (k_j) projections.

Given $x \in \mathrm{FD}(n)$ it is not difficult to see that $x = (g_j \wedge l_j(x)) \vee k_j(x)$.

THEOREM 1. *$x \geqslant y$ if and only if $l_j(x) \geqslant l_j(y)$ and $k_j(x) \geqslant k_j(y)$ for some j.*

PROOF: The necessity is immediate. If, on the other hand, $l_j(x) \geqslant l_j(y)$ and $k_j(x) \geqslant k_j(j)$, then

$$x = (g_j \wedge l_j(x)) \vee k_j(x) \geqslant (g_j \wedge l_j(y)) \vee k_j(y) = y.$$

This shows that every element in FD(n) can be written uniquely as

$$x = (g_j \wedge w) \vee v, \qquad w, v \in \mathrm{FD}(n-1), \quad \text{where} \quad w \geqslant v.$$

Hence, if we call NFD(n) the number of elements of FD(n), and $f_n(x)$ the number of elements of the filter generated by x in FD(n), we obtain

$$\mathrm{NFD}(n) = \sum_{x \in \mathrm{FD}(n-1)} f_{n-1}(x).$$

To obtain a recursive formula for $f_n(x)$ observe that $x \geqslant y$ if and only if $l_j(x) \geqslant l_j(y)$ and $k_j(x) \geqslant k_j(y)$; then

$$f_n(x) = \sum_{\substack{w \geqslant k_j(x) \\ w \in \mathrm{FD}(n-1)}} f_{n-1}(w \vee l_j(x))$$

for any fixed j.

Fixed Points for the Anteriorder Isomorphism

Given $x \in \mathrm{FD}(n)$, x can be written as

$$x = \bigwedge_k \Big(\bigvee_{g_j \in S_k \subset G} g_j \Big)$$

define

$$x^* = \bigvee_k \Big(\bigwedge_{g_j \in S_k \subset G} g_j \Big).$$

The mapping $*$: $\mathrm{FD}(n) \to \mathrm{FD}(n)$ is an anti-isomorphism. The fixed points of such a mapping are those x such that $x = x^*$.

THEOREM 2. *Let* $x \in \mathrm{FD}(n)$, (*write* $x = (g \wedge u) \vee v$; $g \in G$; $u, v \in \mathrm{FD}(n-1)$ *as before*) $x = x^*$ *if and only if* $v = u^*$ $(u \geqslant v)$.

PROOF: If $x = (g \wedge w) \vee w^*$, using that $g = g^*$ and $y^{**} = y$ for every $y \in \mathrm{FD}(n)$:

$$x^* = (g \vee w^*) \wedge w = (g \wedge w) \vee (w \wedge w^*) = (g \wedge w) \vee w^* = w.$$

To prove the converse, observe that

$$l_j(x^*) = (k_j(x))^*$$

and since $x = x^*$ (pick $g = g_1$), $l_1(x^*) = l_1(x) = (k_1(x))^*$, calling $w = l_1(x)$, the theorem follows.

Hence if we define $\mathrm{F}(n) = \{x; x \in \mathrm{FD}(n); x = x^*\}$ and $\mathrm{NF}(n)$ the number of elements of $\mathrm{F}(n)$; Theorem 2 shows that $\mathrm{NF}(n)$ coincides with the number of elements of $\{w, w \in \mathrm{FD}(n-1); w \geqslant w^*\}$.

THEOREM 3. Let $w \in \mathrm{FD}(n)$, $w \geqslant w^*$, if and only if

$$l_j(w) \geqslant k_j(w) \vee (k_j(w))^*, \qquad \text{for some } j.$$

PROOF: $w \geqslant w^*$ if and only if $l_j(w) \geqslant l_j(w^*) = (k_j(w))^*$ and $k_j(w) \geqslant k_j(w^*) = (l_j(w))^*$. Since $l_j(w) \geqslant (k_j(w))^*$ implies $k_j(w) \geqslant (l_j(w))^*$; and since $l_j(w) \geqslant k_j(w)$; then $l_j(w) \geqslant k_j(w) \vee (k_j(w))^*$ and the theorem is established.

Therefore $w \geqslant w^*$; $w \in \mathrm{FD}(n-1)$ (pick $g_j = g_2$) if and only if $w = (g_2 \wedge w_1) \vee v$ where $w_1 \geqslant v_1 \vee v_1^*$; $w_1, v_1 \in \mathrm{FD}(n-2)$. This shows the following formula:

$$\mathrm{NF}(n) = \sum_{x \in \mathrm{FD}(n-2)} f_{n-2}(x \vee x^*).$$

With this formula it is easy to compute that

$\mathrm{NF}(1) = 1$,
$\mathrm{NF}(2) = 2$,
$\mathrm{NF}(3) = 4$,

$\mathrm{NF}(4) = 12,$
$\mathrm{NF}(5) = 81.$

It is also clear from the formula that

$$\mathrm{NF}(n) = \sum_{\substack{x=x^* \\ x\in \mathrm{FD}(n-2)}} f_{n-2}(x) \text{ module } 2$$

In the case $n \equiv 0 \bmod 2$, $\mathrm{FD}(n) \equiv 0 \bmod 2$, but this is essentially due to the fact that in FD(2) there is an anti-isomorphism H such that:

(1) $H(x) \neq x$,
(2) $H^2(x) = H(H(x)) = x$.

Such anti-isomorphism is generated in the following way. Let $h(g_1) = g_2$, $h(g_2) = g_1$, extend h to an isomorphism from FD(2) into itself. Define $H(x) = h(x^*)$.

We shall show that the same result can be extended to any n even.

LEMMA. *Let $G \subset \mathrm{FD}(n)$ as before, $h : G \to G$ any function satisfying*:
(a) $h(g_1) = g_2$; $h(g_2) = g_1$.
Extend h to an homomorphism from $\mathrm{FD}(n)$ *into itself, then* $x = h(x^*)$ *if and only if* (*write* $x = (g_1 \wedge u) \vee v$; $u, v \in \mathrm{FD}(n-1)$)

(1) $l_2(u) = h((f_2(v))^*)$,
(2) $f_2(u) = h(f_2(u)^*)$,
(3) $l_2(v) = h(l_2(v)^*)$,
(4) $f_2(v) = h(l_2(u)^*)$.

PROOF: Observe that from condition (a), for any $y \in \mathrm{FD}(n)$, $l_1(h(y)) = h(l_2(y))$, and $f_1(h(y)) = h(f_2(y))$, write

$$x = (g_1 \wedge u) \vee v, \qquad u \geqslant v,$$
$$x^* = (g_1 \wedge v^*) \vee u^*; \quad h(x^*) = (g_2 \wedge h(v^*)) \vee h(u^*); \quad x = h(x^*)$$

if and only if

(I) $l_1(x) = u = (g_2 \wedge l_1(h(v^*))) \vee l_1(h(u^*))$,
(II) $f_1(x) = v = (g_2 \wedge f_1(h(v^*))) \vee f_1(h(u^*))$.

(I) and (II) will hold if and only if

(1) $l_2(u) = l_1(h(v^*)) = h(l_2(v^*)) = h((f_2 v)^*)$,
(2) $f_2(u) = l_1(h(u^*)) = h(l_2(u^*)) = h((f_2 u)^*)$,

(3) $l_2(v) = f_1(h(v^*)) = h(f_2(v^*)) = h((l_2v)^*)$,

(4) $f_2(v) = f_1(h(u^*)) = h(f_2(u^*)) = h((l_2u)^*)$,

and the lemma follows.

Let now n be even; define $h : G \to G$ as:

$$h(g_{2k-1}) = g_{2k}\,, \qquad k = 1,..., n/2, \qquad h(g_{2k}) = g_{2k-1}\,;$$

extend h to an isomorphism from FD(n) into itself; define $H(x) = h(x^*)$, this an anti-isomorphism in FD(n) and since $h^2(x) = x$, $H^2(x) = x$, we shall show by induction that H leaves no point fixed. The assumption is true for $n = 2$. If n is even, $n > 2$ and $x = H(x) = h(x^*)$, writing $x = (g_1 \wedge u) \vee v$; $u \geqslant v \in \mathrm{FD}(n-1)$ observe that $f_2(u) \in \mathrm{FD}(n-2)$; and applying the previous lemma

$$h((f_2(u))^*) = f_2(u) = H(f_2(u)) \quad \text{in} \quad \mathrm{FD}(n-2), \quad \text{impossible.}$$

Finally, the existence of such an H in FD(n) shows that

$$\mathrm{NFD}(n) \equiv 0 \bmod 2 \quad \text{but} \quad \mathrm{NF}(n) \equiv \mathrm{NFD}(n) \bmod 2.$$

Prime Elements in FD(n)

DEFINITION. $x \in \mathrm{FD}(n)$ is prime if $x = a \vee b$ implies $x = a$ or $x = b$. Using the upper and lower projection technique, it is not difficult to show that $x \in \mathrm{FD}(n)$ is prime if and only if

$$x = \bigwedge_{g_i \in S} g_i \qquad \text{for any set} \quad S \subset G.$$

Hence the prime elements form with the induced order a Boolean algebra where the generators of FD(n) coincide with the maximal elements of the algebra. Every element FD(n) can be identified to a sequence of elements in a Boolean algebra with n generator, where every two elements of the sequence are non-comparable. If we call $V_n{}^r$ the number of such sequences with r elements; and call

$$c_n = \binom{n}{[n/2]} = \frac{n!}{[n/2]!\,(n - [n/2])!}\,,$$

where $[n/2]$ denotes integer part of $n/2$, then

$$\mathrm{NFD}(n) = \sum_{r=1}^{c_n} V_n{}^r.$$

It is possible to show that

$$V_n^1 = 2^n,$$

$$V_n^2 = 2^{n-1}(2^n + 1) - 3^n,$$

$$V_n^3 = \frac{1}{3} \sum_{p=1}^{n-1} \binom{n}{p} \{V_2{}^p(2^{n-p} - 1)^2 + (3^p - 2^{p+1} - 1) \\ \times [2^{n-p}(2^{n-p} - 1) - (3^{n-p} - 1)] + V_{n-p}^2(2^p - 1)\}.$$

A recursive formula for V_n^r will be of great value.

Reference

1. G. Birkhoff, Lattice Theory, *Amer. Math. Soc. Colloq. Publ.* **25** (1948).